THE PIGEONHOLE PRINCIPLE

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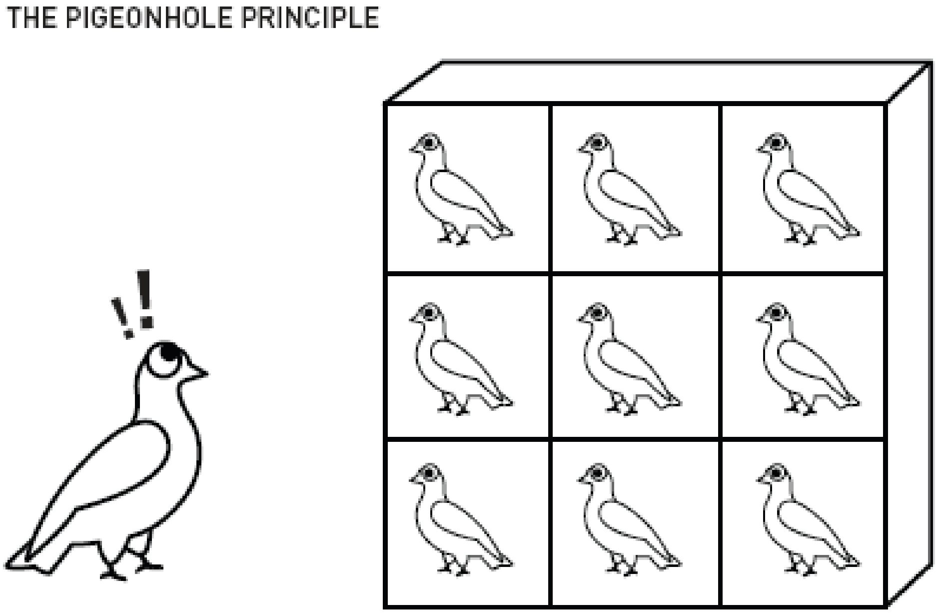
The Pigeonhole Principle: If $n+1$ objects are placed into n boxes, then some box contains at least 2 objects.

Proof: Suppose that each box contains at most one object. Then there must be at most n objects in all. But this is false, since there are $n+1$ objects. Thus some box must contain at least 2 objects.

This combinatorial principle was first used explicitly by Dirichlet (1805-1859). Even though it is extremely simple, it can be used in many situations, and often in *unexpected* situations. Note that the principle asserts the *existence* of a box with more than one object, but does not tell us anything about which box this might be.

In problem solving, the difficulty of applying the pigeonhole principle consists in figuring out which are the 'objects' and which are the 'boxes'.



Problem 1. Prove that among 13 people, there are two born in the same month.

Solution. There are $n = 12$ months ('boxes'), but we have $n+1 = 13$ people ('objects'). Therefore two people were born in the same month.

Problem 2. In Mathville, no person has more that 20000 hairs on their head.

- **(a):** If 20000 people live in Mathville, is it certain that two inhabitants have the same number of hairs on their head?
- **(b):** What is another person moves into Mathville?

Solution.

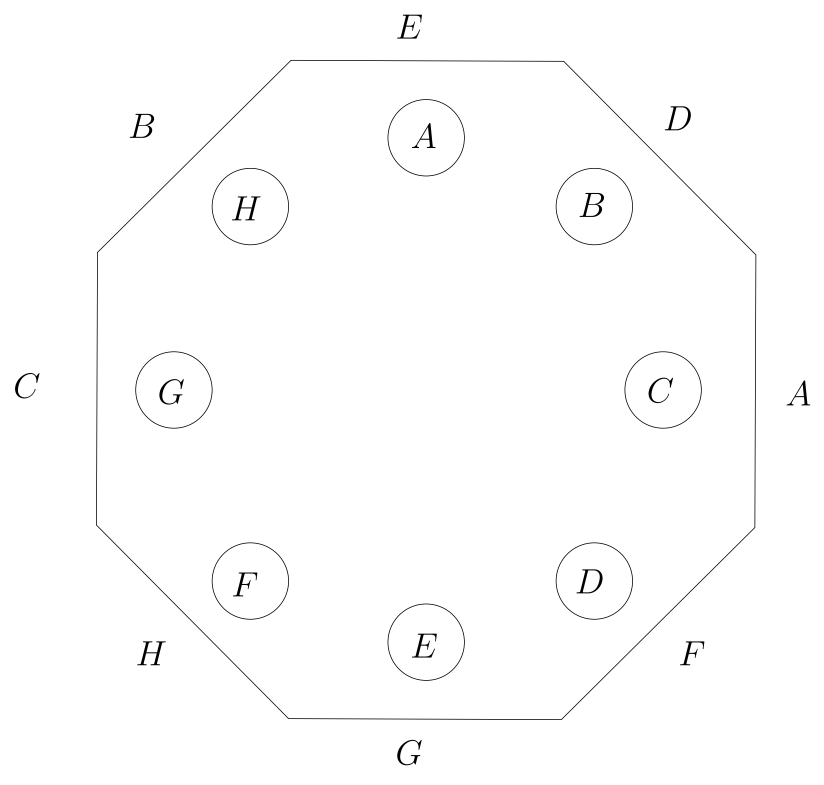
(a): The answer is no. It's possible that the number of hairs on people's heads in Mathville are $1, 2, 3, \ldots, 20000$.

(b): The answer is still no. It's possible that the number of hairs on people's heads in Mathville are $0, 1, 2, 3, \ldots, 20000$.

Question. How many people do you need to be able to say with certainty that two have the same birthday?

Problem 3. There are 8 guests at a party and they sit around an octagonal table with one guest at each edge. If each place at the table is marked with a different person's name and initially everybody is sitting in the wrong place, prove that the table can be rotated in such a way that at least 2 people are sitting in the correct places.

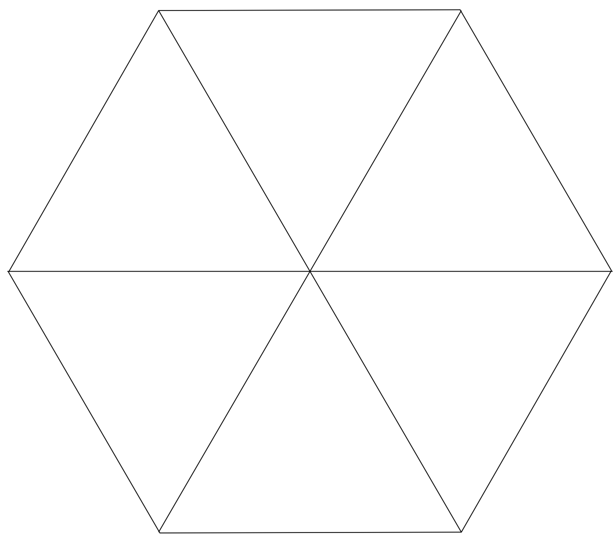
Solution. A typical arrangement is shown below, where the people are marked A, B, C, D, E, F, G, H, and the placenames are marked in circles. In this example, everybody is sitting in the wrong place; for example, guest E is sitting in guest A's place.



For each guest seated around the table, consider that person's *distance to their name* (measured, let's say, clockwise around the table). Since each guest is sitting in the wrong place, the possible distances are $\{1, 2, 3, 4, 5, 6, 7\}$. So while there are 8 guests, there are only 7 possible distances. Therefore by the pigeonhole principle, two guests have the same distance (clockwise) to their name. So rotating the table anticlockwise through this distance will ensure that both of these guests are seated in the correct places. As an illustration, notice that in the picture above, guests D and F are both at distance 2 from their correct positions, so rotating the table 2 places anticlockwise will seat them both correctly.

Problem 4. Seven points lie inside a hexagon of side length 1. Show that two of the points whose distance apart is at most 1.

Solution. Partition the hexagon into six parts as shown below. Now there are six parts (boxes), into which seven points (objects) are distributed. So some part contains at least 2 points. These points must be within distance 1 of each other.



Problem 5. Suppose we have 27 *distinct* odd positive integers all less than 100. ['Distinct' means that no two numbers are equal]. Show that there is a pair of numbers whose sum is 102. What if there were only 26 odd positive integers?

Solution. There are 50 positive odd numbers less than 100:

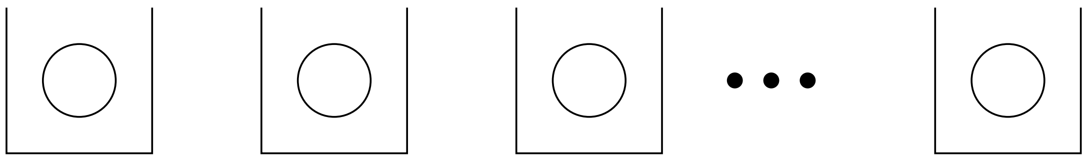
$$\{1, 3, 5, \cdots, 99\} .$$

We can partition these into subsets as follows:

$$\{1\}, \{3, 99\}, \{5, 97\}, \{7, 95\}, \{9, 93\}, \cdots, \{49, 53\}, \{51\}.$$

Note that the sets of size 2 have elements which add to 102. There are 26 subsets (boxes) and 27 odd numbers (objects). So at least two numbers (in fact, exactly two numbers) must lie in the same subset, and therefore these add to 102.

Note on the pigeonhole principle: What if n objects are placed in n boxes? Well, then we *cannot* assert that some box contains at least 2 objects. But note that the only way this can be avoided is if *all* of the boxes contain *exactly one* object.



Problem 6. There are n people present in a room. Prove that among them there are two people who have the same number of acquaintances in the room.

Solution. Each person may have between 0 and $n-1$ acquaintances (inclusive). We imagine labelling each person with the number of acquaintances that person has. We have n people, and n possible values for the labels. We would like to show that some two people have the same label value. If there were more people than label values, we would be finished. But since there is the same number of label values as people, we appear to be stuck.

However, observe that the only way that no two people have the same label value is that everyone has a different label. Thus one person knows nobody, one person knows 1 person, and so on, and finally one person knows $n-1$ people. But this last person then knows everyone else, and in particular this means that there cannot be a person who knows nobody. This contradiction shows that there must indeed be two people who have the same number of acquaintances in the room.

Problem 7. Given 101 positive integers less than 201, prove that there are two of them with the property that one divides the other.

Solution. The following sets will work as pigeonholes:

$$\mathcal{S}_1 = \{1, 2, 4, 8, 16, 32, 64, 128\}$$

$$\mathcal{S}_3 = \{3, 6, 12, 24, 48, 96, 192\}$$

$$\mathcal{S}_5 = \{5, 10, 20, 40, 80, 160\}$$

$$\mathcal{S}_7 = \{7, 14, 28, 56, 112\}$$

$$\mathcal{S}_9 = \{9, 18, 36, 72, 144\}$$

$$\vdots$$

$$\mathcal{S}_{99} = \{99, 198\}$$

$$\mathcal{S}_{101} = \{101\},$$

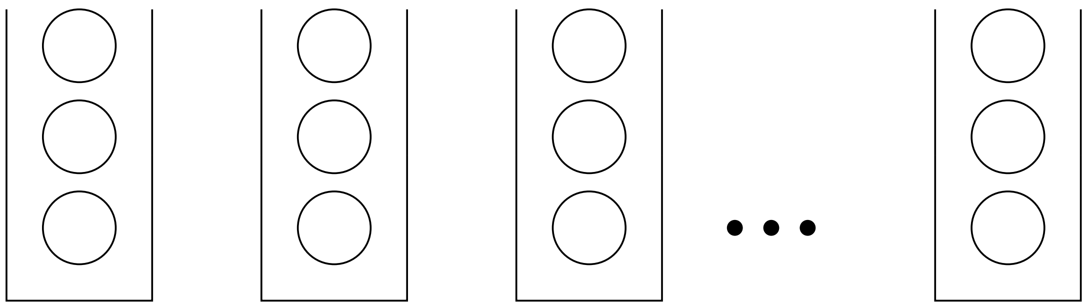
$$\mathcal{S}_{103} = \{103\},$$

$$\vdots$$

$$\mathcal{S}_{199} = \{199\},$$

There are 100 sets (pigeonholes) whose disjoint union is $\{1, 2, \ldots, 201\}$, but we have to choose 101 numbers. So we are sure to choose two numbers from the same set. These two numbers will have the property that one divides the other.

Note on the generalized pigeonhole principle: What if kn objects are placed in n boxes? This means that we cannot assert that some box contains at least $k+1$ objects. But note that the only way this can be avoided is if *all* of the boxes contain *exactly* k objects.



The "Generalized" Pigeonhole Principle: If $kn+1$ objects are placed in n boxes, then some box contains at least $k+1$ objects.

Proof: Suppose that each box contains at most k objects. Then there must be at most kn objects in all. But this is false, since there are $kn+1$ objects. Thus some box must contain at least $k+1$ objects.

Problem 7. Show that in a group of 15 people, at least three were born on the same day of the week.

Solution. We have $15 = 2(7)+1$ people (objects), and 7 weekdays (boxes). Here $k = 2$, $n = 7$. Therefore three people were born in the same day of the week.

Question. How many people do you need to be able to assert with certainty that three have the same birthday?

In-class problems:

Problem. Prove that among any seven square numbers, there are two that end in the same digit.

Problem. 49 counters, each of which can be red, blue or green, are placed into four boxes. Prove that there is a box that contains at least five counters of the same colour.

Problem.

- **(a):** There are 24 seats in a row. Prove that if 17 people sit down, then three consecutive seats will be occupied. Is this true for 16 people?
- **(b):** Find the maximum number of people that can be seated in a row of 25 seats without three consecutive seats being occupied.
- **(c):** Find the maximum number of people that can be seated in a row of 26 seats without three consecutive seats being occupied.

Homework Exercises:

Exercise 1. If ten points lie within a circle of diameter 5, prove that the distance between some two of the points is less than 2.

Exercise 2. Six points lie inside a rectangle of dimensions 3×4. Show that two of the points are at most a distance $\sqrt{5}$ apart.

Exercise 3. 101 girls and 101 boys stand in a circle. Prove that there is a person both of whose neighbours are girls.

Exercise 4. Seven boys and five girls are seated (in an equally spaced fashion) around a table with 12 chairs. Prove that there are two boys sitting opposite each other.

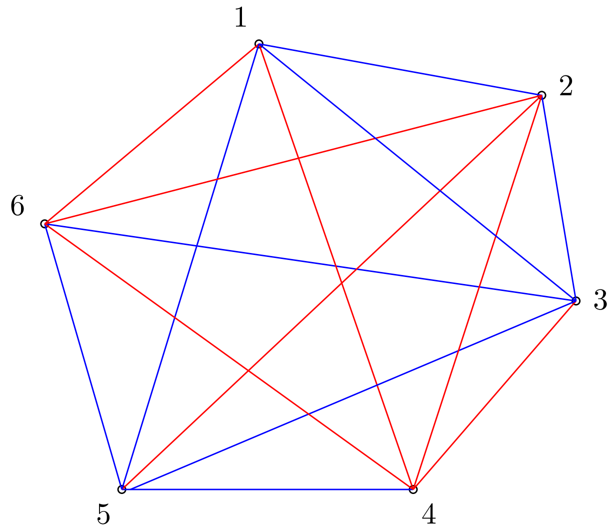
Exercise 5. Each square of a 3×7 board is coloured black or white. Prove that, for any such colouring, the board contains a subrectangle whose four corners are the same colour.

Exercise 6. Prove that however one selects 55 distinct integers from the set $\{1, 2, 3, \ldots, 100\}$, there will be a pair that differ by 9, a pair that differ by 10, a pair that differ by 12, and a pair that differ by 13. Show also that (surprisingly!) there need not be a pair of numbers that differ by 11.

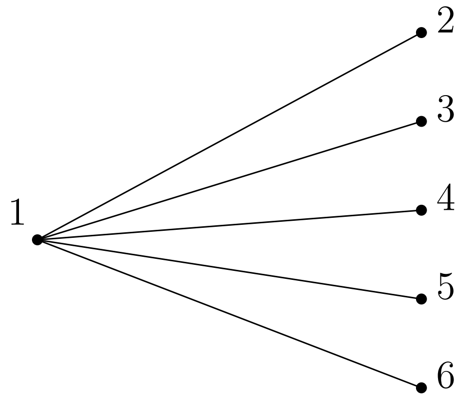
Additional example:

Problem. In any group of six people, prove that there are either 3 mutual friends or 3 mutual strangers.

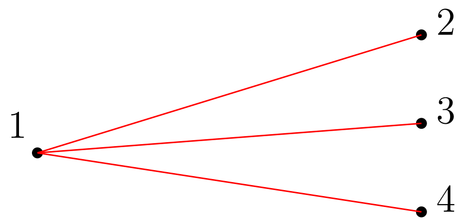
Solution. We can draw a diagram for this problem as follows. Represent the six people by six points in space labelled $1, 2, 3, 4, 5, 6$, and we draw a red edge connecting two points if those people are friends, and a blue edge connecting them if they are strangers. Thus each pair of points is connected by either a red or blue line. We wish to prove that in this configuration, there exists a triangle all of whose edges are the same colour. An example of an edge labelling is shown below; in this example, 146 is a red triangle.



Label the points $1, 2, 3, 4, 5, 6$, and let 36 denote the edge connecting point 3 and point 6, etc. To solve this problem, we begin by considering all the edges emanating from point 1.



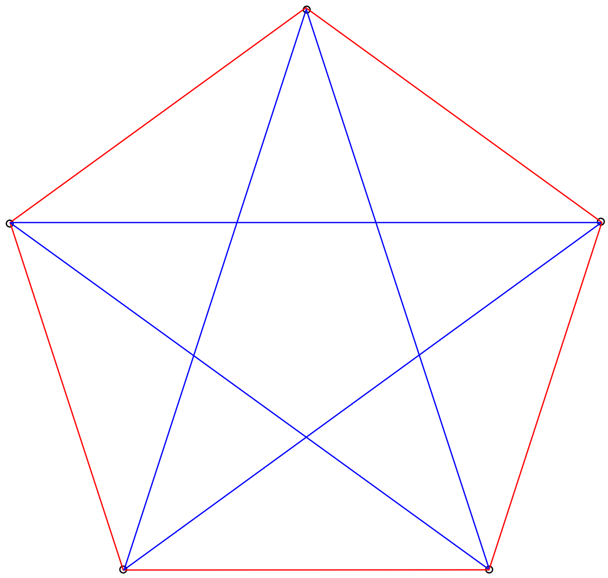
There are 5 of these, but only two colours to paint them, red or blue. Therefore the pigeonhole principle guarantees that at least 3 of them have the same colour. [Here we have $k = 2$ and $n = 2$, i.e. $kn + 1 = 5$ objects placed in $n = 2$ boxes.] Suppose, without loss of generality, that 12, 13 and 14 all have the same colour, let's say they are all red.



If any of 23, 24, 34 is red then we have a red triangle; for example, if 23 is red then the triangle 123 is red. Thus we may assume that 23, 24, 34 are all blue.

But then the triangle 234 is a blue triangle! Therefore there must exist a triangle all of whose edges are the same colour.

Note that with 5 points, it is possible to colour all the edges red or blue without creating a 'monochromatic' triangle; such a colouring is shown below.



This problem belongs to a whole class of related combinatorial problems called **Ramsey Theory**. To read more about this area, click here: Ramsey's Theorem

Exercise. [Difficult.] In the previous problem, prove that you can actually find *two* monochromatic triangles!

[Hint to get you started: From the previous problem we know that there is *one* monochromatic triangle. Suppose without loss of generality that 123 is a red triangle. Now consider the pair of edges 15 and 25. Can anything be said regarding these?]